partial differential equations

types of partial equations

<u>Heat Equation</u>

$$\frac{\partial u}{\partial t} = v\,\frac{\partial^2 u}{\partial x^2}$$
v=constant

- Describes the temp distribution in a thin rod
- The 3D equation $\frac{\partial u}{\partial t} = v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ describes the distribution in 3D objects
- Aka. Diffusion Equation

<u>Wave Equation</u>

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

- Describe the movement / vibration (of a string) / propagation of signals (in a coaxial cable)
- In 3D the equation; $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$ describes *electromagnetic waves* and *water waves*.

~These equations are <u>linear</u>, <u>homogeneous</u>, and <u>second order</u>.
~They have <u>constant coefficients</u>
:. We can add two solution to get another

separation of variables

Heat Equation

1. <u>Need to know initial temperature distribution, and temperature boundaries.</u>

$u(x,0) = 1$ *initial condition*
$u(0,t) = u(l,t) = 0$ *boundary conditions*

2. <u>Principal of Superposition</u>

$$\frac{\partial u_1}{\partial t} = v\,\frac{\partial^2 u_1}{\partial x^2}$$
$$\frac{\partial u_2}{\partial t} = v\,\frac{\partial^2 u_2}{\partial x^2}$$
(add these two equations…)
$$\frac{\partial (u_1+u_2)}{\partial t} = v\,\frac{\partial^2 (u_1+u_2)}{\partial x^2}$$
($u_1 + u_2$ is also a solution… need to look for solutions that we can add in such a way that the boundary and initial conditions will <u>both</u> be satisfied)
$$let\ u(x,t) = X(x)T(t)$$
$$\frac{\partial u}{\partial t} = X\dot{T}$$
$$\frac{\partial^2 u}{\partial x^2} = X''T$$

$$X\dot{T} = vX''T$$
$$\frac{1}{v}\frac{\dot{T}}{T} = \frac{X''}{X} = k$$
k = constant of separation

3. <u>Ordinary Differential Equations</u>

$$\dot{T} - vkT = 0$$
$$X'' - kX = 0$$

4. <u>'k' – 3 Cases</u>

I. **k>0**
let $k = q^2$ $\quad X(x) = Ae^{qx} + Be^{-qx}$

<u>boundary conditions</u>
$X(0) = 0 \rightarrow A + B = 0$
$X(l) = 0 \rightarrow Ae^{ql} + Be^{-ql} = 0$
$A = B = 0$
:. Solution is trivial :. No good.

II. **k=0**
$$X(x) = Ax + B$$
$A = B = 0$
:. Solution is trivial :. No good.

III. **k<0**
let $k = -p^2$ $\quad X(x) = Acospx + Bsinpx$

<u>boundary conditions</u>
$X(0) = 0 \rightarrow A = 0$
$X(l) = 0 \rightarrow Bsinpl = 0$
:. Solution is non-trivial :. Good!
$$\sin pl = 0$$
$$pl = n\pi$$
$$p = \frac{n\pi}{l}$$
$$k = -p^2 = -\frac{n^2\pi^2}{l^2}$$

$$\boldsymbol{X_n = B_n \sin\frac{n\pi x}{l}}$$

5. <u>Use k value to find corresponding value of T_n</u>

$$\dot{T} - vkT = 0$$
$$\dot{T}_n + \frac{vn^2\pi^2}{l^2}T_n = 0$$
$$\dot{T}_n + \lambda_n^2 T_n = 0$$
So $\boldsymbol{T_n = A_n e^{-\lambda_n^2 t}}$

6. <u>Sub values for Xn and Tn back into first u(x,t) equation</u>
$$u(x,t) = X(x)T(t)$$
$$\boldsymbol{u_n(x,t) = B_n \sin\frac{n\pi x}{l} e^{-\lambda_n^2 t}}$$

7. <u>Satisfy Initial Conditions</u>
$$u(x,t) = \sum_n u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\, e^{-\lambda_n^2 t}$$
$$u(x,0) = 1 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}$$

I. <u>Use ½ Range Fourier Series to find B_n for the function $f(x) = 1$</u>
$$B_n = \frac{2}{l}\int_0^l \sin\frac{n\pi x}{l}\,dx = \begin{cases} \frac{4}{n\pi}, & (n\ odd) \\ 0, & (n\ even) \end{cases}$$

8. <u>Plug Bn value into sum equation + work out series</u>
$$u(x,t) = \sum_n u_n(x,t)$$
$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{l}\,e^{-\frac{(2n-1)^2\pi^2}{l^2}t}$$

<u>Note:</u> as $t \rightarrow 0, u \rightarrow \infty$. For large values of t, the temp is roughly proportional to $\sin\frac{\pi x}{l}$, as this part of the solution decays least rapidly.

Wave Equation

1. <u>Initial and Boundary Conditions</u>

$u(x,0) = f(x)$ *initial conditions*
$u_t(x,0) = g(x)$
$u(0,t) = u(l,t) = 0$ *boundary conditions*

2. <u>*Principal of Superposition*</u>
$$let\ u(x,t) = X(x)T(t)$$

3. <u>Ordinary Differential Equations</u>

$$\ddot{T} - c^2kT = 0$$
$$X'' - kX = 0$$

4. <u>'k' – 3 Cases</u>
$$k = -\frac{n^2\pi^2}{l^2}$$
$$\boldsymbol{X_n(x) = \sin\frac{n\pi x}{l}}$$

5. <u>Use k value to find corresponding value of T_n</u>
$$\ddot{T} - \lambda_n^2 T = 0 \qquad \lambda_n = \frac{cn\pi}{l}$$
$$T_n = A_n cos\lambda_n t + B_n sin\lambda_n t$$

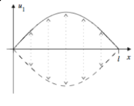
6. <u>Sub values for Xn and Tn back into first u(x,t) equation</u>
$$u_n(x,t) = \sin\frac{n\pi x}{l}\,(A_n cos\lambda_n t + B_n sin\lambda_n t)$$

7. <u>Satisfy Initial Conditions</u>
$$u(x,t) = \sum_n u_n(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\,(A_n cos\lambda_n t + B_n sin\lambda_n t)$$

i. $u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$
→hence, A_n is the coefficient of the ½-rang Fourier sine series of f(x)

ii. $u_t(x,t) = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\,(-A_n\lambda_n sin\lambda_n t + B_n\lambda_n cos\lambda_n t)$
$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n\lambda_n \sin\frac{n\pi x}{l})$$
→hence, $B_n\lambda_n$ is the coefficients of the ½-rang Fourier sine series of g(x)

<u>Note</u>: each term in the series for u(x,t) is a harmonic of the motion



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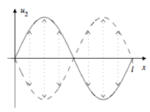

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