vector calculus

divergence of a vector field

<u>Divergence</u> of a vector field: if the flux of a vector field, $\hat{F}$, over a surface, *s,* is the amount of $\hat{F}$ pointing out of the volume.

If *F* represents the velocity of a fluid then flux would be the rate at which the fluid flows out of the volume

If we divide the flux by the outflow rate /unit volume:

$$\frac{1}{V}\int\int \hat{F}\cdot d\hat{s}$$

*If we do this over a very small volume δV, we can determine whether fluid is flowing towards or away from a point this quantity is called the **divergence**.*

$$\lim_{\Delta V\to 0}\frac{1}{\delta V}\int\int_S \hat{F}\cdot d\hat{S}$$

$$\delta V = \delta x\delta y\delta z$$
$$F = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$$
$$near\ the\ point\ (x,y,z)$$

TOTAL FLUX:

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\delta x\delta y\delta z$$

FLUX DENSITY aka **DIVERGENCE**:

$$\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$
$$\left(\hat{\imath}\frac{\partial}{\partial x}+\hat{\jmath}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(u\hat{\imath}+v\hat{\jmath}+w\hat{k}\right) = \nabla\cdot F$$

- As divergence = flux density, it follows that if we integrate the divergence over a volume, we expect to get the total flux for that volume (or the surface of that volume). Thus;

$$\int\int\int_v \nabla\cdot F\, dV = \int\int_s F\cdot dS$$

(Integral of flux density) *(Total Flux)*

Divergence Theorem aka <u>Gauss' Theorem</u>

- Divergence Theorem leads to an important identity in fluid mechanics…
- If $F = \rho v$, then $\int\int_s \rho v\ \cdot n\, dS$ is the <u>rate</u> at which fluid leaves the volume enclosed by *S*.

  The total <u>mass</u> in this volume is $\iint\int_v \rho\, dV$, so the <u>rate of change of the mass</u> is $\frac{d}{dt}\iint\int_v \rho\, dV$.

  Therefore;

$$\frac{d}{dt}\iint\int_v \rho\, dV = -\int\int_s \rho v\ \cdot n\, dS$$
$$\therefore \int\int\int_v \frac{\partial\rho}{\partial t}\, dV = -\int\int\int_v \nabla\cdot(\rho v)\, dV$$

(Using the divergence theorem)

$$\therefore \int\int\int_v \left(\frac{\partial p}{\partial t}+\nabla\cdot(\rho v)\right) dV = 0$$

$$\therefore \frac{\partial\rho}{\partial t}+\nabla\cdot(\rho v) = 0 \quad everywhere$$

Continuity Equation

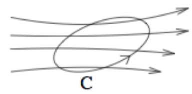
(expresses the conservation of mass)

<u>Note</u>: if the fluid is incompressible, then ρ is a constant and the equation becomes $\nabla\cdot B = 0$. Such fields are called *solenoidal.*

curl of a vector field

<u>Curl</u> of a vector field: effectively the tendency of a fluid to rotate about a point.

To calculate place a closed loop, *C*, in the fluid, in a plane normal to the given direction and calculate $\int_c F\ \cdot dr$ around the closed curve. This is the ***circulation*** of <u>F</u> around *C*.



Circulation Density

$$\lim_{\delta S\to 0}\frac{1}{\delta S}\int_c F\ \cdot dr$$

Curl F

$$\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\hat{\imath}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\hat{k}$$

$$\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}$$

$$\nabla\times F$$

<u>Example</u>

If $F = -xy\hat{\imath} + z^2\hat{\jmath} - yz\hat{k}$, calculate $\nabla\times F$.

$$\nabla\times F = \begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -xy & z^2 & -yz\end{vmatrix}$$
$$= (-z-2z)\hat{\imath} - (0-0)\hat{\jmath} + (0+x)\hat{k}$$
$$= -3z\hat{\imath} + x\hat{k}$$

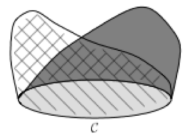
Note: each component of $\nabla\times F$ is like a circulation/unit area or a *circulation density*.

If we integrate the density over some surface *S,* we expect to get back the total circulation around the boundary of the surface. That is;

$$\int\int_s (\nabla\times F)\cdot n\, dS = \int_c F\ \cdot dr$$

Stokes Theorem

C is the boundary curve of the surface S



If *C* is fixed and we attempt to calculate $\int\int_s(\nabla\times F)\cdot n\, dS$ over a surface bounded by *C*, we get the same answer no matter what surface we choose.

some important identities

If $f(x,y,z)$ and $g(x,y,z)$ are scalar functions and F and G are vector fields the following identities hold;

$$\nabla(f+g) = \nabla f + \nabla g$$
$$\nabla(F+G) = \nabla\cdot F + \nabla\cdot G$$
$$\nabla\times(F+G) = \nabla\times F + \nabla\times G$$

<u>Product Rules</u>

$$\nabla(fg) = f\nabla g + g\nabla f$$
$$\nabla\cdot(fG) = \nabla f\cdot G + f\nabla\cdot G$$
$$\nabla\cdot(F\times G) = (\nabla\times F)\cdot G - F\cdot(\nabla\times G)$$
$$\nabla\times(fG) = \nabla f\times G + f\nabla\times G$$

<u>Second Derivatives</u>

$$\nabla\cdot(\nabla f) = \nabla^2 f$$
$$\nabla\times(\nabla f) = 0$$
$$\nabla\cdot(\nabla\times F) = 0$$

Note: $\nabla\times G = 0$ if and only if

- there is a scaler function *f(x,y,z)* such that $G = \nabla f$. In this case G is <u>conservative</u> or <u>irrotational.</u>
- there is a vector field F such that $G = \nabla\times F$. In this case G is <u>incompressible</u> or <u>solenoidal</u>.